Corollary to Green's Theorem (Ch 16.4, Pg 1099)

(This is used on number 16.4.21 in the homework.)

First, recall that Green's Theorem gave us (where D is enclosed by C),

$$\oint_C Pdx + Qdy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

with appropriate assumptions on the curve C and on P and Q.

Since $Area(D) = \iint\limits_D 1dA$, we can get the following from going backwards with Green's Theorem,

Corollary. *If P and Q satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then we can recover the area of the domain D by*

$$Area(D) = \iint\limits_D 1dA = \oint_C Pdx + Qdy.$$

Notably, some common ways to make $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ are

$$\begin{cases} 1. \quad Q(x,y) = x, \quad P(x,y) = 0. \\ 2. \quad Q(x,y) = 0, \quad P(x,y) = -y. \quad (\textit{take heed of the minus sign!}) \\ 3. \quad Q(x,y) = \frac{x}{2}, \quad P(x,y) = -\frac{y}{2} \quad (\textit{take heed of the minus sign!}) \end{cases}$$

and these give the expressions for the area, (take heed of minus signs and the $\frac{1}{2}$ factor from 3.)

$$Area(D) = \oint Pdx + Qdy = \begin{cases} \oint_C xdy & \textit{from 1.} \\ \oint_C -ydx & \textit{from 2.} \\ \frac{1}{2} \oint_C -ydx + xdy & \textit{from 3.} \end{cases}$$

(All three ways are equivalent and will get the Area correctly).

Sometimes it's more convenient to compute this line integral over C instead of the double integral over D. For example, see Example 3 on Pg 1099. Hopefully Prof. Banerjee got to it.

If we were to do Example 3 as a double integral, we would have to do a change of variables to make the ellipse into a circle (as a quick question, what would the new variables u, v be?), and then integrate. Here, with Green's Theorem, we can just do a line integral.

On the homework problem, it is using form (3) above for making $Q_x - P_y = 1$.